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Origin of Exponential Solution for Laminar Decay of Isolated Vortex

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The origin of the exact, closed-form, self-similar exponential solution for the laminar decay of an isolated two-dimensional vortex in a viscous incompressible fluid is studied to find a proper name for the function. Two major contributions to the solution are the similarity parameter derived by Boltzmann and the derivation of the form of the solution by Carslaw for the diffusion of heat from a line source, that is, a spark or lightning stroke. Lamb adapted the heat transfer solution derived by Carslaw to the time-dependent diffusion of vorticity from a potential line vortex, and, after publication of his book, has provided a short, complete, and readily available derivation. Oseen's derivation followed about four years later and is not as complete. Various titles for the solution are believed to be appropriate, including a generic name such as exponential vortex solution, or the name of Lamb. The name of Lamb is probably the most appropriate because he was the first in existing literature to put the entire derivation together and because the presentation is compact, complete, and readily available.

Nomenclature

r = radius, ft (m)
t = time, s
v_θ = swirl velocity, ft/s (m/s)
Γ = circulation, ft^2/s (m^2/s)
ζ = vorticity, $(1/r)(\partial v_\theta/\partial r)$
η = $r^2/4\nu t$
μ = viscosity, ft^2/s (m^2/s)
ν = μ/ρ
ρ = air density, slug/ft^3 (kg/m^3)

Subscripts

c = vortex core
0 = total

I. Introduction

IN scientific research, it is fairly common practice to name a process, an equation, or a solution after the person that originated it, for example, Navier–Stokes equations, Mach number, Reynolds number, etc. Because guidelines for the naming process have apparently not been developed, it is a somewhat random process that can vary over a wide range of naming techniques. On some occasions, the name of the originator is immediately associated with the solution so that the source remains correctly identified. At other times, the solution is not identified with the originator, but is assigned some other common name, and an association with the originator is lost. The purpose of the study reported here is to find out who was original author of the exponential solution for the decay of an isolated vortex that was introduced about 100 years ago and then determine the most appropriate title or name for the solution.

The exponential solution for the decay of an isolated vortex was used extensively after the mid-1900s while an intense research effort was directed at finding one or more aerodynamic modifications for subsonic transport aircraft that would dramatically reduce the hazard posed by their lift-generated wakes. It was reasoned that a substantial reduction in the rolling-moment hazard posed by vortex wakes would permit in-trail spacings between aircraft to be safely reduced during landing and takeoff operations at airports. To date, an implementable solution has not been found. The research carried out, however, did produce a large amount of information on the structure and decomposition of vortex wakes of aircraft.[1] Much of the information obtained was guided by the exponential solution for the decay of an isolated vortex because it provided a simple and exact solution in closed form for the time- and radially dependent laminar decay of vortices. The solution also provided a functional form for the interpretation of experimental results and for the evaluation of numerical methods. An illustration of the change in appreciation that has taken place for the subject solution is apparent in the content of various editions of Schlichting's book, that is, the first edition (1951) makes no mention of this exact solution, but a later edition (1968) does include the solution along with some information on it.[2]

The origin of the exponential solution for the decay of an isolated vortex is discussed in this paper to be better able to recommend an appropriate name for this very useful function. Uncertainty exists because a clearly presented derivation for the vortex solution appears to not have been presented in any one of the documents that could be considered as the origin. This paper first presents an overview of the derivation of the vortex solution. A description is then given on the analytical techniques used to solve problems in fluid mechanics about 100 years ago and on the origin of the exponential vortex solution. A proper name for the solution is then discussed.

II. Overview of Solution

The search during the late 1800s and early 1900s for exact analytical solutions to the Navier–Stokes equation were successful when the flowfields were correctly idealized to match the experimental ones used for comparison (see Ref. 3). The closed-form solution for the exponential decay of a vortex is a special case of solutions available from the idealized Navier–Stokes and continuity equations wherein the time-dependent flowfield is axially symmetric, and time dependent.[4–10] The differential equation for the swirl velocity in such an idealized flowfield is usually given by

$$\rho\frac{\partial v_\theta}{\partial t} = \mu\left[\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2}\right] \tag{1a}$$

which can be reduced to

$$\rho\frac{\partial v_\theta}{\partial t} = \mu\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial r v_\theta}{\partial r}\right) \tag{1b}$$

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where ρ is the density of the fluid in which the vortex is embedded, v_θ is the swirl velocity at a time t, and radius r from the center of the axially symmetric vortex. Note that, because the reduced differential equation is linear, superposition of solutions is permissible.

The initial boundary conditions on the flowfield specify that the distribution of swirl velocity at the beginning of the event, when $t = 0$, is given by the structure of a point potential vortex given as $v_\theta(r, 0) = \Gamma_0/2\pi r$, where Γ_0 is the total circulation in the vortex, and r is the radius from the concentrated circulation at the center of the flowfield. As time increases from zero, the circulation, originally concentrated at $r = 0$, diffuses into the flowfield to yield realistic distributions of vorticity and swirl velocity that often approximate measured ones. As such, the swirl velocity is singular at the beginning of the decay process. Thereafter, the radial distribution of the vorticity is given by

$$v_\theta(r, t) = (\Gamma_0/2\pi r)[1 - \exp(-r^2/4\nu t)] \tag{2}$$

where $\nu = \mu/\rho$ is the kinematic viscosity of the fluid in which the vortex is embedded. The radius at which the maximum swirl velocity occurs, that is, at the core radius r_c of the vortex, is given by $r_c^2/4\nu t = 1.25643121$.

It is apparent from Eq. (2) that, if the circulation $\Gamma(r, t)$ is substituted for the quantity $2\pi r v_\theta(r, t)$, the solution may be written as

$$\Gamma(r, t) = \Gamma_0[1 - \exp(-r^2/4\nu t)] \tag{3}$$

which is a function of only the one parameter, $\eta = r^2/4\nu t$, which makes it self-similar. Carslaw[5,6] attributes the origin of the parameter η to Boltzmann,[4] who recommended its use as a way to simplify the solution of differential equations of diffusion type. Had Boltzmann been aware of the need for a solution to the laminar decay of a vortex, he could probably have very easily written down the solution given by Eq. (2) or (3).

At about the same time, Rayleigh and others noted that vorticity, like heat, is a property of the fluid. The analogous differential equation and boundary conditions in terms of vorticity are then the same as the ones used to study the diffusion of heat, thereby providing a useful overlap in analytical solutions that can be used in both disciplines.[3] The differential equation written in terms of the swirl velocity v_θ is now written in terms of the vorticity given by $\zeta = (1/r)(\partial v_\theta/\partial r)$ as

$$\frac{\partial \zeta}{\partial t} = \nu\left[\frac{\partial^2 \zeta}{\partial r^2} + \frac{1}{r}\frac{\partial \zeta}{\partial r}\right] \tag{4}$$

which is identical with the differential equation for the radial flow of heat in two dimensions.[5–7] The solution for vorticity is then given by

$$\zeta(r, t) = (\Gamma_0/4\pi r \nu t)\exp(-r^2/4\nu t) \tag{5}$$

which, like v_θ, is not purely a function of η. The vorticity is related to the circulation by

$$\Gamma(r, t) = \int_0^r 2\pi r \zeta \,\mathrm{d}r \tag{6}$$

The expression for $\zeta(r, t)$ was used by Oseen[9,10] to develop a Green's function for use in finding a category of general solutions to the time-dependent Navier–Stokes equations. It appears that these quite idealized formulations were never put into common use, probably because they did not properly represent the self-induced movement of the vortical flowfield.

III. Status of Science Around 1900

It is difficult for most of us to comprehend how research was conducted in the late 1800s, which was at a time before most buildings had electric lights and before electronic computers. It is even more amazing how communication and transportation have changed in one lifetime from primarily a horse-and-buggy mode to the now common air and space travel along with a variety of electronic communications. As late as the early 1900s, most computations were carried out by hand multiplication and division. In addition, the openly quite hostile, aggressive, and isolationist attitudes of the major nations in the world toward each other did much to impede communication of scientific results across borders. As a consequence, it is not surprising that parallel efforts and results often occurred independently in a way that led to identification of different names with essentially the same result, for example, it is not uncommon today for some to refer to the Navier–Stokes equations as the Navier equations and others as the Stokes equations.

One consequence of the lack of electricity and electronic computers was the requirement placed on the derived analytical solutions. That is, the solutions had to be simple enough that computations for various applications could be carried out by use of tabulated functions and hand computations. Therefore, it was necessary for analysts to idealize the problem being studied enough so that the formulas derived would provide usable results. In addition, the idealizations made in the theory must be realistic enough that the solutions still represented the flowfield with enough fidelity to be a reliable representation of the actual problem. Both of these requirements are met by the exponential solution for a decaying vortex.

Before the time that work began on finding solutions to the flowfields produced by various vortex structures, much of this type of effort had been devoted to the study of Kelvin's idea that atoms are vortex rings in a fluid, that is, vorton structure of matter (see Refs. 3, 9, and 10). In an effort to examine the viability of the idea, considerable study appears to have been devoted to the growth and decay of vortex rings, and other vortical flowfields, to find out how rapidly the atomic models would disappear if the fluid were not perfectly inviscid, but had a small viscosity. When the vorton theory of matter was replaced by the nucleus/electron model, the investigations turned to practical vortical flowfields found in laminar and turbulent flowfields that could be readily observed and measured.

The study of vortices was often based on the observation of turbulent eddy-type motions in fluids. Because vortex eddies in a viscous fluid do slow down or decay with time, it was necessary for the simulation to contain time-dependent models of vortices. The effort devoted to simulation of the eddy-type motions observed in turbulent flowfields led to the development of representations of isolated and multiple vortex structures embedded within flowfields.[3] It was out of this work that the more elaborate and general representations of vortical flowfields were eventually idealized to a single isolated vortex and its decay with time.

IV. Citation/Derivation History

A. Background Functions

Two parameters or functions, namely, η and ζ, play an important role in the development of solutions to the diffusion equations. As mentioned previously, Carslaw[5,6] and Carslaw and Jaeger[7] attribute the origin of the parameter η to Boltzmann.[4] In his paper, an explanation or derivation of the parameter is not given. It simply contains a recommendation for their use and shows how such a parameter can be used to simplify the solution of differential equations of diffusion type by combining the radius and time into one independent variable. In the literature reviewed, a number of articles also utilized the function ζ without mentioning a reference source. However, the 1906 date of Carslaw's first book appears to be early enough that it was the first time that the parameter was derived from a complex series type of function to yield a very simple closed-form solution for a heat transfer problem, namely, the transfer or diffusion of heat from a spark or lightning stroke as a function of time. If the derivation had been made some tens of years after the presentation made in Carslaw's 1906 book, it probably would have been done by means of the Laplace transform method, which is a much simpler route than the one that appears to have been used by Carslaw.[5] The Laplace transform was not used because Oseen and Carslaw both point out that Heaviside's method (forerunner of the Laplace transform method) is too cumbersome and intuitive for them to use, and the newly developed method is not yet established well enough for use (see Refs. 5, 10, and 11).

Instead, the derivation is carried out along the path in common use during the late 1800s and early 1900s, known then as the Fourier analysis method and now known as the separation of variables. The process first expresses the differential equations $v_\theta(r, t)$ as the product of two functions; one a function only of radius r and the other only a function of time t. The solution found is then the product of the two solutions derived separately from the two ordinary differential equations. The two functions are coupled through special characteristic values. Because the problem is linear, a general solution is constructed as the sum or series of these products (an infinite series of solutions), wherein the terms are added together to fulfil the boundary conditions for the event being simulated. When the physical field of interest (flow of fluid or heat) is one dimensional, the functions are exponentials in time, and distance is expressed by sine and cosine functions. If the physical field of interest is axially symmetric, the solution for the spatial dimension r is expressed by Bessel functions, which are much more cumbersome to use in a computation than sine or cosine functions. Batchelor[11] provides a good description of the foregoing process.

It is the conversion of the infinite series of solutions expressed by the exponential- and Bessel-function products that were summed by Carslaw[5] to arrive at the ζ and exponential functions used by Lamb and Oseen for application to the flowfield of decaying vortices as expressed by Eqs. (2) and (3). The analytical route used by Carslaw[5] begins with the same differential equation as given by Eq. (1a) and separates the solution into two parts as described in the preceding paragraph. After some manipulations of the functions, Carslaw proceeds to sum the resulting series/integral by means of the relationship

$$\int_{-\infty}^{\infty} \exp(-\nu t a^2) \cos ax \, \mathrm{d}a = \sqrt{\frac{\pi}{\nu t}} \exp\left(\frac{-x^2}{4\nu t}\right) \tag{7}$$

thereby providing a solution for the heat transfer and viscous-vortex flowfields. Interestingly, the foregoing integral connection of the closed-form solution to the Fourier–Bessel series solution is only described by Carslaw in the 1906 edition of his book. All other editions simply state that the solution to the differential equation is available from the literature on the corresponding heat transfer problem, and a citation to a source, or to the origin, of the solution is not mentioned.

Oseen[9,10] appears to be one of the most, if not the most, prolific user of the special functions η and ζ, and the only reference found to be cited by him for the origin of these parameters is to simply state that they are available out of the theory for heat transfer solutions. It makes one wonder whether all of the literature on the subject of diffusion (and it is a large amount) goes back to the integral relationship quoted by Carslaw in his 1906 book.[6] If so, the solutions obtained by Oseen and by Lamb were both obtained by the same path, namely, they refer to the literature on the diffusion of heat for the solution to the differential equation to be solved. Similarly, in the thorough review of the literature of the time on analytical solutions for fluid mechanics by Dryden, Murnaghan, and Bateman,[3] they also always refer to the solution found by heat transfer theory without giving a citation. Lamb and Batchelor both refer to the 1921 edition of Carslaw's book,[6] which refers to but does not contain the derivation. The correct reference for the derivation is found only in the 1906 edition of Carslaw's book.

B. Exponential Vortex Solution

Oseen uses the exponential solution for decaying vortices in a number of his papers, but it is not until publication of his book on hydrodynamics in 1927 that he presents a derivation for the formula, without a reference to its origin.[9,10] In that derivation he, as did Lamb, derives the governing equations and demonstrates the virtues of the solution, but refers to the literature on heat transfer theory for the solution to the differential equation, which is the hard part. Because copies of these original publications are difficult to obtain for most researchers working on the decomposition and decay of vortices, users of the solution have referred to Lamb's book[8] because it is readily available, or to Oseen's paper in 1911 as the source, which does not contain the derivation. In the book written by Lamb, the differential equation for the flowfield is set up and solved. The text then attributes its solution to Carslaw.[6] However, the correct reference for the solution to the differential equation is Carslaw's 1906 book.[5]

Oseen's 1911 paper[9] does not contain the derivation of Eq. (2) or (3), but is devoted to the derivation of criteria for when the model for decaying vortices provides the same motion velocities as provided by Helmholtz's inviscid rules, that is, when the diffusion of vorticity does not cause vortical regions to overlap, which does not take into account the effect of shearing velocities on the structure and redistribution of the vorticity contained in the vortices.

Interestingly, none of the literature points out that a simple, straightforward path to the same solution is available by use of the similarity parameter, $\eta = r^2/4\nu t$, which reduces the partial differential equation to an ordinary differential equation that is easy to solve. It seems truly amazing that so many participants in these theoretical studies did not recognize the simplest path to the same solution that was essentially suggested by Boltzmann.[4] That is, if the partial differential equation (1b) is transformed into an ordinary differential equation by use of the similarity parameter η, it becomes the ordinary differential equation

$$\rho \frac{\mathrm{d}\Gamma}{\mathrm{d}\eta} = \frac{\mathrm{d}^2\Gamma}{\mathrm{d}\eta^2} \tag{8}$$

which can be integrated. When the boundary conditions described earlier are applied, the solution is again given by Eq. (3). The originator of this technique is uncertain even though the parameter η has been known in the literature at least since Boltzmann recommended its use for solving problems of the diffusion type. A solution through substitution into the differential equations appears to not have been carried out by any of the early researchers who might have served as an originator of the exponential solution for vortex decay.

V. Conclusions

The original objective of the study was to find the correct origin for the simple exact solution for the time-dependent decay of an isolated vortex. It was found that none of the names associated with the derivation of the exponential solution for the decay of an isolated vortex have a clearly defined derivation path from beginning to end. The best available reference and description of the derivation is presented by Lamb.[8] The citation by Lamb clearly indicates how to arrive at a solution and then points out that the solution to the differential equation comes from Carslaw. Although fairly complete as written, Lamb should have referred to the 1906 version of Carslaw's book[5] (which is really hard to obtain) instead of the 1921 edition.[6]

Thus, how should the exponential solution for the decay of an isolated vortex be called? It is the writer's opinion that the best name is the one that calls the solution by its physical process, namely, the exponential solution for a decaying vortex. If a name is to be attached to the solution, it should not be Oseen's and could be either Lamb's or Carslaw's. Because Carslaw treated only the heat transfer problem, that is, diffusion of heat from a spark or lightning stroke, and not the fluid dynamics vortex decay, association of his name with the decaying vortex solution does not seem quite appropriate. It is Lamb who extended Carslaw's solution to the vortex flowfield and then provided aerodynamicists with a solution for vortex decay. If a descriptive title is preferred, a complete version is given by the exact, closed-form, self-similar exponential solution for the laminar decay of an isolated two-dimensional vortex in a viscous incompressible fluid. Of various titles proposed, exponential solution for a decaying vortex, or Lamb's solution, or a combination of the two appear to be most appropriate. Of the various possibilities, it is recommended that the title Lamb's solution for a decaying vortex be chosen because Lamb was the first to present a compact and complete derivation that is still readily available to most readers worldwide.

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